Normal confidence interval

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$$\hat{p} \underset{appox}{\sim} N\left(p, \frac{p(1-p)}{n}\right) \Rightarrow U \approx \frac{\hat{p}-p}{\sqrt{p(1-p)/n}} \sim N(0,1) \Rightarrow$$

$$z_{\frac{\alpha}{2}} \le U \approx \frac{\hat{p}-p}{\sqrt{p(1-p)/n}} \le z_{1-\frac{\alpha}{2}} \quad ((1-\alpha)\% \text{ confidence interval of } U) \Rightarrow$$

$$p \approx \hat{p} \pm \frac{z_{\frac{\alpha}{2}}}{\sqrt{n}}\sqrt{\hat{p}(1-\hat{p})}$$ （这里用 z_α 替代 $z_{1-\frac{\alpha}{2}}$）

注：当n较小、或p的值靠近0或1时，由此得到的置信区间会出现overshoot或zero-width的情况。

置信区间的上/下限超过1或小于0

置信区间宽为0，这实际是 falsely implying certainty

对于给定的sample size n，可以解
$$\begin{cases} \hat{p}+\frac{z_\alpha}{\sqrt{n}}\sqrt{\hat{p}(1-\hat{p})} > 1 \\ \hat{p}-\frac{z_\alpha}{\sqrt{n}}\sqrt{\hat{p}(1-\hat{p})} < 0 \end{cases}$$
得到当 $\hat{p}$ 处于什么范围时会出现overshoot的情况。

e.g. $\hat{p}+\frac{z_\alpha}{\sqrt{n}}\sqrt{\hat{p}(1-\hat{p})} > 1 \Rightarrow 1-\hat{p} < \frac{z_\alpha}{\sqrt{n}}\sqrt{\hat{p}(1-\hat{p})} \Rightarrow (1-\hat{p})^2 < \frac{z_\alpha^2}{n}\hat{p}(1-\hat{p}) \Rightarrow$

$\frac{1-\hat{p}}{\hat{p}} < \frac{z_\alpha^2}{n} \Rightarrow \frac{1}{\hat{p}} < 1+\frac{z_\alpha^2}{n} \Rightarrow \hat{p} > \frac{1}{1+\frac{z_\alpha^2}{n}}$

$\hat{p}-\frac{z_\alpha}{\sqrt{n}}\sqrt{\hat{p}(1-\hat{p})} < 0 \Rightarrow \hat{p}^2 < \frac{z_\alpha^2}{n}\hat{p}(1-\hat{p}) \Rightarrow \frac{\hat{p}}{1-\hat{p}} < \frac{z_\alpha^2}{n} \Rightarrow \frac{1-\hat{p}}{\hat{p}} > \frac{n}{z_\alpha^2} \Rightarrow$

$\frac{1}{\hat{p}} > 1+\frac{n}{z_\alpha^2} \Rightarrow \hat{p} < \frac{1}{1+\frac{n}{z_\alpha^2}}$

总的说来，normal confidence interval 只适用于 n 很大 且 p 靠近 0.5 的情况，否则会出现 overshoot/zero-width 的情况，最重要的是，所构建的置信区间的 coverage probability 远小于 $1-\alpha$，也就不是 $(1-\alpha)\%$ confidence interval 了。

Wilson score interval

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基于二项分布的正态近似，我们有

$z_\alpha \approx \frac{p-\hat{p}}{\sigma_n}$, $\sigma_n = \sqrt{\frac{p(1-p)}{n}}$ $\Rightarrow$ $(1+\frac{z_\alpha^2}{n})p^2 - (2\hat{p}+\frac{z_\alpha^2}{n})p + \hat{p}^2 = 0$ $\Rightarrow$

其解为 p 的 $1-\alpha$ 置信区间，而非 p 的估计值。$\Rightarrow$

$$p \in_{\approx \alpha} (w^-, w^+) = \frac{1}{1+z_\alpha^2/n}\left(\hat{p} + \frac{z_\alpha^2}{2n} \pm \frac{z_\alpha}{2n}\sqrt{4n\hat{p}(1-\hat{p})+z_\alpha^2}\right)$$

$\hookrightarrow P\{p \in (w^-, w^+)\} = 1-\alpha$

对于上述 confidence interval，我们能够保证 the average coverage probability is $1-\alpha$。

若要保证 the minimum coverage probability is $1-\alpha$，我们可以使用 the Wilson score interval with continuity correction:

$$w_{cc}^- = \max\left\{ 0 \,,\ \frac{2\,n\,\hat{p} + z_\alpha^2 - \left[z_\alpha \sqrt{z_\alpha^2 - \frac{1}{n} + 4\,n\,\hat{p}\,(1-\hat{p}) + (4\,\hat{p}-2)} \;+ 1 \right]}{2\,(n+z_\alpha^2)} \right\},$$

$$w_{cc}^+ = \min\left\{ 1 \,,\ \frac{2\,n\,\hat{p} + z_\alpha^2 + \left[z_\alpha \sqrt{z_\alpha^2 - \frac{1}{n} + 4\,n\,\hat{p}\,(1-\hat{p}) - (4\,\hat{p}-2)} \;+ 1 \right]}{2\,(n+z_\alpha^2)} \right\},$$

for $\hat{p} \neq 0$ and $\hat{p} \neq 1$.

If $\hat{p} = 0$, then w_{cc}^- must instead be set to 0 ; if $\hat{p} = 1$, then w_{cc}^+ must be instead set to 1 .

Clopper-Pearson interval

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The Clopper-Pearson interval is often called an "exact" method, as its coverage probability is never less than the nominal $1-\alpha$.

Based on the relationship between the binomial distribution and the beta distribution, we can represent the Clopper-Pearson interval as:

$$B\left(\frac{\alpha}{2}; k, n-k+1\right) < p < B\left(1-\frac{\alpha}{2}; k+1, n-k\right)$$

Note: $B(p; v, w)$ is the pth quantile from a beta distribution with shape parameters v and w.